Integration by Method of [illegible]
(Problems)

Algebraic integrand :-

Form : $\int \frac{dx}{\sqrt{ax+b} \pm \sqrt{ax+c}}$

Problem (1)

$$\int \frac{dx}{\sqrt{5-3x} + \sqrt{4-3x}}$$

$$= \int \frac{\sqrt{5-3x} - \sqrt{4-3x}}{5-3x-4+3x}\, dx$$

$$= \int (5-3x)^{1/2} dx - \int (4-3x)^{1/2} dx$$

using $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$$= \frac{(5-3x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-3)} - \frac{(4-3x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-3)}$$

$$= -\frac{2}{3} \frac{(5-3x)^{3/2}}{3} + \frac{2}{3} \frac{(4-3x)^{3/2}}{3}$$

$$= -\frac{2}{9}[5-3x]^{3/2} + \frac{2}{9}(4-3x)^{3/2} + C$$

Algebraic standard integral :-

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$\int \frac{1}{x} dx = \log|x|$

$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b|$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$

$\int \frac{1}{1+x^2} dx = \tan^{-1} x$

$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$

Form :- $\int f(x) [g(x)]^n dx$ When $g(x)$ is in standard form

Method :→ Say

$$f(x) = A\, g(x) + B$$

$$\therefore f(x)[g(x)]^n = [A\, g(x) + B] [g(x)]^n$$

$$= A[g(x)]^{n+1} + B[g(x)]^n$$

$$\therefore \int f(x)[g(x)]^n dx = A \int [g(x)]^{n+1} dx + B \int [g(x)]^n dx$$

for value of A and B

$f(x) = A\, g(x) + B$

put $x=0$ and $x=1$ and

then solve for A and B

Ex: → $\int (x+2)\sqrt{x-3}\, dx$

solution Here → $f(x) = x+2$ $[g(x)]^{1/2} = [x-3]^{1/2}$

put $f(x) = A\, g(x) + B$ —①

$\therefore (x+2)(x-3)^{1/2} = [A(x-3) + B](x-3)^{1/2}$

$= A(x-3)^{3/2} + B(x-3)^{1/2}$

$\therefore \int (x+2)(x-3)^{1/2} dx = A\int (x-3)^{3/2} dx + B\int (x-3)^{1/2} dx$

using $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$= \frac{2A}{5}(x-3)^{5/2} + \frac{2}{3}B(x-3)^{3/2} + C$

$= 2(x-3)^{3/2}\left[\frac{A}{5}(x-3) + \frac{1}{3}B\right] + C$ —①

for A and B

put $x=0$ and $x=1$ successively in ①

$f(0) = A\, g(0) + B$

$f(1) = A\, g(1) + B$

$\therefore f(1) - f(0) = A[g(1) - g(0)]$

$\therefore A = \frac{f(1) - f(0)}{g(1) - g(0)}$

$= \frac{3-2}{-2+3} = 1$

Here $f(x) = x+2$

$\Rightarrow f(0) = 2$

$f(1) = 3$

$g(x) = x-3$

$g(0) = -3$

$g(1) = -2$

$\therefore B = 2 - A\, g(0)$

$= 2 - 1\cdot(-3) = 5$

From (i)

$\int (x+1)\sqrt{x-3}\,dx$

$= 2(x-3)^{3/2}\left[\frac{x-3}{5}+\frac{4}{3}\right]+C$

$= \frac{2}{15}(x-3)^{3/2}\left[3x-9+25\right]+C$

$= \frac{2}{15}(x-3)^{3/2}(3x+16)+C$

<u>Trigonometric Transformation</u>

<u>Standard Trigonometric Integral</u>

$\int \sin x\,dx = -\cos x$, $\int \cos x\,dx = \sin x$, $\int \sec x\tan x\,dx = \sec x$

$\int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x$ $\int \sec^2 x\,dx = \tan x$

$\int \operatorname{cosec}^2 x\,dx = -\cot x$

1. $\int \cos^2 x\,dx$; $\cos^2 x = \frac{1}{2}(\cos 2x+1)$

Solution - $I = \int \cos^2 x\,dx$

$= \frac{1}{2}\int [1+\cos 2x]\,dx$

$= \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]$

$= \frac{x}{2}+\frac{\sin 2x}{4}+C$

2. $I = \int \sec^2 x\operatorname{cosec}^2 x\,dx$

$= \int \frac{1}{\sin^2 x\cos^2 x}\,dx$; $\sin x\cos x = \frac{1}{2}\sin 2x$

$\Rightarrow \sin^2 x\cos^2 x = \frac{1}{4}\sin^2 2x$

$= \int \frac{1}{\frac{1}{4}\sin^2 2x}\,dx$

$= 4\int \operatorname{cosec}^2 2x\,dx = -\frac{4\cot 2x}{2}+C = -2\cot 2x+C$

Integrals

$$\int \cos x \cos 2x \cos 3x \, dx$$ use $\cos A \cos B = \frac{1}{2}\{\cos(A+B) +$

$$= \int [\cos 2x \cos x] \cos 3x \, dx$$

$$= \frac{1}{2}\int [\cos 3x + \cos x] \cos 3x \, dx$$

$$= \frac{1}{2}\int [\cos^2 3x + \cos 3x \cos x] \, dx$$

$$\frac{1}{2}\int \left[\frac{1+\cos 6x}{2} + \frac{1}{2}(\cos 4x + \cos 2x)\right] dx$$

$$\frac{1}{4}\int [1 + \cos 6x + \cos 4x + \cos 2x] \, dx$$ use $\int dx =$

$\int \cos kx$

$=$

$$\frac{1}{4}\left[x + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right] + C$$

(iii) $\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)} dx$

$x-\alpha = x+\alpha - 2\alpha$

$\sin(x-\alpha) = \sin\{(x+\alpha) - 2\alpha\}$

$= \sin(x+\alpha)\cos 2\alpha - \cos(x+\alpha)\sin 2\alpha$

$\therefore \frac{\sin(x-\alpha)}{\sin(x+\alpha)} = \cos 2\alpha - \cot(x+\alpha)\sin 2\alpha$

$\therefore I = \int \frac{\sin(x-\alpha)}{\sin(x+\alpha)} dx$

$= \int [\cos 2\alpha - \cot(x+\alpha) \cdot \sin 2\alpha] dx$

$= \cos 2\alpha \cdot x - \sin 2\alpha \log \sin(x+\alpha) + C$

<u>Evaluate</u> $I = \int \frac{\cos(x+a)}{\sin(x+b)} dx$

$\because x+a = (x+b) + (a-b)$

$\therefore \cos(x+a) = \cos\{(x+b) + (a-b)\}$

$= \cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)$

Now $\frac{\cos(x+a)}{\sin(x+b)} = \frac{\cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)}{\sin(x+b)}$

$= \cos(a-b)\cot(x+b) - \sin(a-b)$

$\therefore I = \int \frac{\cos(x+a)}{\sin(x+b)} dx = \cos(a-b)\int \cot(x+b)dx - \sin(a-b)\int dx$

$= \cos(a-b)\log \sin(x+b) - \sin(a-b) \cdot x + C$

use.

$[\because \int \cot(ax+b)dx = \frac{1}{a}\log \sin(ax+b)]$

Evaluate $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$

$$I = \int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$$

Since $\sin x = \cos(\frac{\pi}{2} - x)$
$= \sin x = \cos(\pi/2 + x)$
$\Rightarrow \sin x = -\cos(\pi/2 + x)$

$$= \int \tan^{-1}\sqrt{\frac{1 + \cos(\pi/2 + x)}{1 - \cos(\pi/2 + x)}}$$

$\sqrt{\frac{1+\cos x}{1-\cos x}} = \sqrt{\cot^2 \frac{x}{2}}$

$\sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\tan^2 \frac{x}{2}}$

$$= \int \tan^{-1}\left[\tan\left(\pi/4 - \frac{x}{2}\right)\right]dx$$

$\sin^{-1}(\sin x) = x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\cos^{-1}(\cos x) = x$, $0 \le x \le \pi$

$\tan^{-1}(\tan x) = x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$= \int \left[\frac{\pi}{4} - \frac{x}{2}\right]dx$$

$$= \frac{\pi}{4}\cdot x - \frac{x^2}{4} + c$$

$$= \frac{\pi}{4}x - \frac{x^2}{4} + c$$

Evaluate $I = \int \cot^{-1}(\sec x + \tan x)\,dx$

$$= \int \cot^{-1}\left[\frac{1+\sin x}{\cos x}\right]dx$$

$$= \int \cot^{-1}\left[\frac{(\cos\frac{x}{2} + \sin x/2)^2}{\cos^2 x/2 - \sin^2 x/2}\right]dx$$

$$= \int \cot^{-1}\left[\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right]dx$$

$= \int [\pi/4 + x/2]\,dx = \int \cot^{-1}[\tan(\pi/4 - x/2)]\,dx = \int \cot^{-1}[\cot[\frac{\pi}{2} - \frac{\pi}{4} + \frac{x}{2}]]$